On Source Analysis by Wave Splitting with Applications in Inverse Scattering of Multiple Obstacles

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November 27, 2006

Abstract

We study wave splitting procedures for acoustic or electromagnetic scattering problems. The idea of these procedures is to split some scattered field into a sum of fields coming from different spatial regions such that this information can be used either for inversion algorithms or for active noise control.

Splitting algorithms can be based on general boundary layer potential representation or Green's representation formula. We will prove the unique decomposition of scattered wave outside the specified reference domain G and the unique decomposition of far-field pattern with respect to different reference domain G. Further, we employ the splitting technique for field reconstruction for a scatterer with two or more separate components, by combining it with the point source method for wave recovery. Using the decomposition of scattered wave as well as its far-field pattern, the wave splitting procedure proposed in this paper gives an efficient way to the computation of scattered wave near the obstacle, from which the multiple obstacles which cause the far-field pattern can be reconstructed separately. This considerably extends the range of the decomposition methods in the area of inverse scattering. Finally, we will provide numerical examples to prove the feasibility of the splitting method.

Keywords. Inverse scattering, wave splitting, potential theory, near field, regularization, numerics.

1 Introduction

Inverse problems for acoustic and electromagnetic waves play an important role in many scientific and engineering applications. Medical imaging for example uses several techniques from the area of inverse problems as basic ingredients for medical

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examinations. Nondestructive testing employs inverse problems techniques for quality control. For a given incident wave, the impenetrable obstacle D will generate a scattered wave outside D, which is in general governed by the Helmholtz equation for acoustic waves or Maxwell equations for electromagnetic waves. The scattered wave and its far-field pattern contain information about the scatterer D such as the boundary shape and boundary type. The reconstruction of an obstacle D from the far-field pattern of its scattered wave is one of the central research topics in inverse scattering theory, see for example [10] and the topical review [13].

There are three categories of shape reconstruction methods from far-field data of scattered waves. Firstly, there are iterative schemes, compare [5]. The second kind of methods use decomposition and optimization techniques, which firstly determine the scattered wave from its far-field pattern on a set outside of the obstacle and then update this surface such that the total field matches the boundary data by iteration procedure. This is a classical method with a long history [5, 7, 10]. The third kind of methods, which have been developed recently, constructs some indicator function of the boundary from the near field or its far-field pattern, respectively. Then the boundary shape is constructed from the point set where the indicator blows up in some way [2, 3, 4, 14, 15]. In most of these methods, the reconstruction of the scattered wave from its far-field pattern is of great importance.

By expressing the scattered wave outside D in the form of a potential integral defined on ∂D, the direct scattering problem determines the scattered wave and its far-field using a density function which satisfies an integral equation on ∂D. This procedure is also applicable if D contains multiple connected components ([16]). Different methods for reconstructing the scattered field outside of D have been developed, compare the literature given in [5]. Here we will employ and further develop the *potential method* of Kirsch-Kress [5] and the *point source method* of Potthast, Erhard, Liu, Chandler-Wilde, Lines and others (see [1], [10]). The methods reconstruct the scattered or total fields, respectively, outside of some auxilliary domain G. Both methods in their standard formulation have problems with reconstructing multiple obstacles $D = \bigcup D_j$ when the location of the obstacles is not known a-priori.

Following the potential approach, in principle we can still choose an approximate domain G satisfying $G \supset \overline{D}$ such that we can compute the scattered wave outside G. The choice of G can be specified from the knowledge of far-field pattern via *range test method* [12], where the solvability of some integral equation is tested (compare Section 2.4). However, it is a complex algorithmical task to use multiple auxilliary domains G to reconstruct the field close to the boundary shapes ∂D of the unknown scatterer D. Also, this leads to severe stability problems, since the ill-posedness of the equations under consideration depends on the curvature and non-convexity of the curves [8, 11].

Motivated by these problems, we present an efficient way to reconstruct the scattered wave from the far-field pattern caused by multiple obstacles. The basic idea is to *split the far-field pattern* into several parts which are essentially related to each obstacle. Correspondingly, the scattered wave is also decomposed. Please observe that our splitting *avoids any approximation* as for example employed for the Born approximation or physical optics approximation. Using this idea based on general potential theory or Green representation formula and combining it with the *point source method*, we propose a scheme which provides a reconstruction of the scattered wave at all points outside of some scatterer D with several components. This *splitting method* enables the recovery of the scattered wave outside of multiple obstacles. The method proposed in this paper, except for its intrinsic importance in *wave recovery*, is also applicable to *shape reconstruction* for multiple obstacles.

This work is organized as follows. In Section 2, we firstly give the exact definition of scattered wave splitting and prove the *uniqueness* for this splitting for a given domain G. Then we establish two methods for the wave splitting, which are based on a *single-layer* approach and *Green's formula*, respectively. The uniqueness for both approaches is proven. To achieve flexibility in the choice of a reference domain G, we also prove the uniqueness of far-field pattern decomposition with respect to G. Then in Section 3, by using the wave splitting technique developed here together with the point source method, we investigate field and shape reconstructions for an obstacle D containing multiple disjoint connected components. This important application of wave splitting provides a novel and efficient way to the reconstruction of multiple obstacles in the frame work of evaluating the scattered wave. Finally, we show the numerical feasibility of the scheme by presenting some numerical reconstructions in Section 4.

2 Scattered wave splitting

To explain the basic idea of scattered wave splitting for multiple obstacles, here we assume that the obstacle D contains *two* disjoint connected components D_1, D_2, that is, $D = D_1 \cup D_2$ such that $D_1 \cap D_2 = \emptyset$ and D_j is connected with boundary of class C^2 for $j = 1, 2$.

In order to specify the domain where the total scattered wave for the obstacle $D = D_1 \bigcup D_2$ is decomposed, we firstly give

DEFINITION 2.1. *Assume that two domains G_1 and G_2 with $G_1 \cap G_2 = \emptyset$ and C^2-smooth boundary are given such that $\overline{D_1} \subset G_1$ and $\overline{D_2} \subset G_2$. Set $G := G_1 \cup G_2$.*

Denote by $\Phi(\cdot,\cdot)$ the free-space fundamental solution to the Helmholtz equation $\Delta u+\kappa^2 u=0$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. For G given in Definition 2.1, the single- and double-layer potentials are defined by

$$(S\varphi)(x) := \int_{\partial G} \Phi(x,y)\varphi(y)ds(y), \tag{1}$$

$$(K\varphi)(x) := \int_{\partial G} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y) \tag{2}$$

for $x\in\mathbb{R}^m$ and solve the Helmholtz equation in $\mathbb{R}^m\setminus\partial G$. Moreover we introduce

$$(K'\varphi)(x) := 2\int_{\partial G} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)ds(y), \quad x\in\partial G, \tag{3}$$

$$(T\varphi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_{\partial G} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y), \quad x\in\partial G. \tag{4}$$

It is well known the above four integrals called potential functions are well-defined for $x\in\partial G$ with density φ in suitable Hölder or Sobolev spaces (see [5]).

2.1 Uniqueness of source splitting

This section serves to establish the uniqueness of a general scattered field splitting for some domain G given by Definition 2.1. Here, we do not need to specify the concrete form of the potentials under consideration.

THEOREM 2.2. *Consider domains G_j as given in Definition 2.1. Assume that we are given a decomposition $u^s=u_1^s+u_2^s$ of the scattered field u^s such that*

1. *u_j^s satisfies the radiation condition for $j=1,2$;*
2. *u_j^s soles the Helmholtz equation in the exterior of G_j for $j=1,2$;*
3. *Both $(u_j^s)^+$ and $\frac{\partial(u_j^s)^+}{\partial\nu}$ exist in ∂G_j, where*

$$(u_j^s)^+|_{\partial G_j} := \lim_{x\in\mathbb{R}^m\setminus\overline{G}_j,\, x\to\partial G_j} u_j^s(x).$$

Then the splitting of u^s is unique, i.e. for every further splitting $u^s=\tilde{u}_1^s+\tilde{u}_2^s$ with $\tilde{u}_j^s$ meeting conditions 1-3 in this theorem, we obtain $u_j^s(x)=\tilde{u}_j^s(x)$ for $x\in\mathbb{R}^m\setminus\overline{G}_j$ with $j=1,2$.

Proof. We subtract the two representations and use the definition

$$v_j(x) := u_j^s(x) - \tilde{u}_j^s(x),\ x \in \mathbb{R}^m \setminus \overline{G}_j,\ j = 1, 2 \tag{5}$$

to obtain in $\mathbb{R}^m \setminus \overline{G_1 \cup G_2}$ that

$$v_1 + v_2 = (u_1^s + u_2^s) - (\tilde{u}_1^s + \tilde{u}_2^s) = u^s - u^s = 0. \tag{6}$$

Thus, noticing the smoothness of v_1 in the neighbor of G_2, we know that v_1 is a solution to the Helmholtz equation in G_2 with boundary values

$$v_1|_{\partial G_2} = -v_2^+|_{\partial G_2}, \quad \frac{\partial v_1}{\partial \nu}|_{\partial G_2} = -\frac{\partial v_2^+}{\partial \nu}|_{\partial G_2}. \tag{7}$$

We now define the function

$$w := \begin{cases} v_1 & \text{in } \overline{G}_2 \\ -v_2 & \text{in } \mathbb{R}^m \setminus \overline{G}_2. \end{cases} \tag{8}$$

Then the function w solves the Helmholtz equation in $\mathbb{R}^m \setminus \partial G_2$, both its value and normal derivative are continuous on ∂G_2. Therefore w is analytic in $\mathbb{R}^m$ due to the analytic continuation, since v_1, v_2 are analytic function in $G_2, \mathbb{R}^m \setminus \overline{G}_2$ respectively. Thus we have constructed an entire radiating solution w to the Helmholtz equation in $\mathbb{R}^m$, which must vanishes everywhere. We obtain $v_1 \equiv 0$ in G_2 and by analytic continuation it is zero also in the exterior of G_1. The same arguments show $v_2 \equiv 0$ outside G_2, which ends the proof. □

In the following, we will establish the concrete decomposition scheme for scattered wave in terms of its far-field pattern.

2.2 Far field splitting via single-layer potentials

Here, we will describe the splitting of far field pattern u^∞ for scattering of an acoustic wave u^i. Then this splitting generates the desired scattered wave decomposition in $\mathbb{R}^m \setminus \overline{G}$.

Let us express the scattered wave by single-layer approach

$$u^s(x) := (S\varphi)(x), \ \ x \in \mathbb{R}^m \setminus \overline{G}. \tag{9}$$

The far field pattern of $S\varphi$ is given by the operator

$$(S^\infty \varphi)(\hat{x}) := \gamma \int_{\partial G} e^{i\kappa \hat{x} \cdot y} \varphi(y) ds(y), \ \ \hat{x} \in \mathbb{S} \tag{10}$$

with $\gamma = 1/(4\pi)$ in $\mathbb{R}^3$ and $\gamma = e^{i\pi/4}/\sqrt{8\pi\kappa}$ in $\mathbb{R}^2$, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^m$. Here, the density φ lives on $\partial G = \partial G_1 \cup \partial G_2$. We denote

$$\varphi_j(y) := \varphi(y) \text{ for } y \in \partial G_j \tag{11}$$

and denote the corresponding single-layer potential operators by S_j, i.e.

$$(S_j\varphi_j)(x) := \int_{\partial G_j} \Phi(x,y)\varphi_j(y)ds(y), \quad x \in \mathbb{R}^m \tag{12}$$

and we have

$$S\varphi = S_1\varphi_1 + S_2\varphi_2. \tag{13}$$

ALGORITHM 2.3. *The splitting of the far field of a scatterer $D = D_1 \bigcup D_2$ is obtained from the following three steps.*

1. *Solve the far-field equation*

$$S^\infty\varphi = u^\infty \tag{14}$$

   to generate density function φ defined in ∂G, where S^∞ is given via (10).

2. *Define two functions*

$$u_j^s(x) := (S_j\varphi_j)(x), \quad x \in \mathbb{R}^m \setminus \overline{G_j},\ j = 1,2, \tag{15}$$

   which can be considered as a scattered wave outside G_j, in the sense that it solves the Helmholtz equation in $\mathbb{R}^m \setminus \overline{G}_j$ and meets the radiation condition.

3. *Compute the far field patterns of u_j^s defined by*

$$u_j^\infty := S_j^\infty\varphi_j, \quad j = 1,2. \tag{16}$$

In this way, the far field pattern u^∞ is decomposed as

$$u^\infty = u_1^\infty + u_2^\infty. \tag{17}$$

Correspondingly, the scattered wave u^s related to u^∞ has the splitting

$$u^s(x) = u_1^s(x) + u_2^s(x), \quad x \in \mathbb{R}^m \setminus \overline{G} \tag{18}$$

from the linear superposition principle and Rellich lemma, where u_j^s is computed via (15). Moreover, u_j^s outside G_j is the scattered wave related to u_j^∞ with $j = 1,2$ again from Rellich lemma, noticing $u_j^s(x)$ defined by (15) is the radiation solution.

For the feasibility of the above scattered wave splitting based on the far-field pattern decomposition we need to investigate the following questions.

1. Is (14) uniquely solvable? If so, then the decomposition (17) and the functions u_1^s, u_2^s in (15) are uniquely defined.

2. For given G, is (18) a decomposition of u^s in the sense of Theorem 2.2? If this is the case then the single-layer approach is a *constructive method* for this unique decomposition of the scattered field.

THEOREM 2.4. *Assume that G is chosen in the way of Definition 2.1 such that $-\kappa^2$ is not the interior Dirichlet eigenvalue of Δ in G_j for $j = 1, 2$. Then there exists a unique solution $\varphi \in L^2(\partial G)$ to (14).*

Remark. The proof contains two parts. Firstly, we prove that the far field operator $S^\infty : L^2(\partial G) \to L^2(\mathbb{S})$ is injective. This proof is completely the same as that G is a connected domain, compare [5], [7]. However, to understand the advantage of wave splitting based on the Green formula in the next subsection, which does not need the assumption on the wave number κ^2, we still give the proof here. Secondly, we give the existence of φ in $L^2(\partial G)$.

Proof. Let $S^\infty\varphi = 0$ on $\mathbb{S}$. Then the scattered wave u^s expressed by (9) has zero far-field. Therefore it follows from the Rellich lemma that $u^s = (S\varphi)(x) \equiv 0$ outside G. By the continuity of single layer potential on ∂G, we know that $(S\varphi)(x)$ solves the Helmholtz equation in G with boundary condition $(S\varphi)(x) = 0$ in ∂G. Since κ^2 is not the Dirichlet eigenvalue of $-\Delta$ in G_j from the definition of G_j, then we get $(S\varphi)(x) \equiv 0$ in G_j. So we get $(S\varphi)(x) \equiv 0$ in $\mathbb{R}^m \setminus (\partial G_1 \bigcup \partial G_2)$. Using the jump relation of $(S_1\varphi_1)(x)$ and the continuity of $(S_2\varphi_2)(x)$ in ∂G_1, we get $\varphi_1 = 0$. Similarly, $\varphi_2 = 0$. Therefore S^∞ is injective.

For the existence of solution in $L^2(\partial G)$, we refer to the following Theorem 2.10. For its applicability we remark that for the domain G chosen by the Definition 2.1 the scattered wave u^s corresponding to the far field pattern u^∞ naturally has an extension into the exterior of G. □

This result gives an answer to the first equation. Now, we can prove the uniqueness of the decomposition of scattered wave, which may be stated as

THEOREM 2.5. *If u^s has the decomposition (18) outside G in the sense of Theorem 2.2, then u_j^s can be calculated using the single-layer potential as defined in (15).*

Proof. Due to Theorem 2.2, it is enough to prove that u_j^s with $j = 1, 2$ defined in (15) meet the three conditions in Theorem 2.2.

It follows from Theorem 2.4 that u_j^s is well defined outside G_j and meets the condition 1 and condition 2 in Theorem 2.2 obviously. On the other hand, it follows from the following Theorem 2.10 again that u_j^s can be extended analytically to $\mathbb{R}^m \setminus \overline{G}_j$. Therefore condition 3 is also met. □

This result gives a positive answer to the second question. Since the choice of G meeting the previous conditions is not unique, we must consider the uniqueness of far-field decomposition (17) and the scattered wave decomposition for different choice of G. This uniqueness can be stated as

THEOREM 2.6. *Denote by u^∞ the far-field pattern caused by obstacle D and u^s the scattered wave outside $\overline{D}$ related to u^∞. Assume that $\tilde{G} := \tilde{G}_1 \bigcup \tilde{G}_2$ different from G is the other configuration satisfied the same requirement on G given previously. If we decompose the far-field pattern u^∞ as*

$$u^\infty = \tilde{u}_1^\infty + \tilde{u}_2^\infty \tag{19}$$

using the same algorithm given above for $\tilde{G}$ and construct $\tilde{u}_j^s(x)$ outside $\tilde{G}_j$ by the density function $\tilde{\varphi}_j$ related to $\partial\tilde{G}_j$, then we have for $i = 1, 2$ that

$$\tilde{u}_j^\infty(\hat{x}) = u_j^\infty(\hat{x}) \tag{20}$$

and

$$\tilde{u}_j^s(x) = \tilde{u}_j^s(x), \quad x \in \mathbb{R}^m \setminus \overline{G_j \bigcup \tilde{G}_j}, \tag{21}$$

provided that $(G, \tilde{G})$ meets the following separation condition

$$\overline{G_1 \bigcup \tilde{G}_1} \bigcap \overline{G_2 \bigcup \tilde{G}_2} = \emptyset. \tag{22}$$

Proof. We prove this theorem splitting the proof into the following two cases. First, we treat the case where G_j' contains G_j in its interior for both indices $j = 1, 2$. Secondly, we reduce the general case to this special case.

Case 1: $G_j \bigcap \tilde{G}_j = G_j$ with $j = 1, 2$. For given $G, \tilde{G}$, it follows from $\tilde{u}_1^\infty + \tilde{u}_2^\infty = u^\infty = u_1^\infty + u_2^\infty$ that

$$(\tilde{u}_1^\infty - u_1^\infty) + (\tilde{u}_2^\infty - u_2^\infty) = 0. \tag{23}$$

On the other hand, noticing the correspondence between $(\tilde{u}_j^\infty, u_j^\infty)$ and $(\tilde{u}_j^s, u_j^s)$ in terms of the density $(\tilde{\varphi}_j, \varphi_j)$ with $j = 1, 2$, we know that $(\tilde{u}_1^s - u_1^s) + (\tilde{u}_2^s - u_2^s)$ outside $\tilde{G}$ is the scattered wave corresponding to the far-field pattern $(\tilde{u}_1^\infty - u_1^\infty) + (\tilde{u}_2^\infty - u_2^\infty)$, since it is the radiating solution to the Helmholtz equation outside $\tilde{G}$. Therefore the Rellich lemma and (23) yield

$$[(\tilde{u}_1^s - u_1^s) + (\tilde{u}_2^s - u_2^s)](x) \equiv 0, \ x \in \mathbb{R}^m \setminus \overline{\tilde{G}}. \tag{24}$$

Now applying the same argument as that in the proof of Theorem 2.2 with G there replaced by $\tilde{G}$, we get that

$$\tilde{u}_j^s = u_j^s,\ x \in \mathbb{R}^m \setminus \overline{\tilde{G}_j}$$

for $j = 1, 2$, which proves (21), noticing in this case $G_j \bigcup \tilde{G}_j = \tilde{G}_j$. Now using the relation between $(\tilde{u}_j^\infty, u_j^\infty)$ and $(\tilde{u}_j^s, u_j^s)$ in terms of the density $(\tilde{\varphi}_j, \varphi_j)$ again, we know $(\tilde{u}_j^\infty, u_j^\infty)$ are the far-field pattern of scattered wave $(\tilde{u}_j^s, u_j^s)$. Therefore (21) leads to (20) immediately.

Notice, in this case, our proof does not need the condition (22), which is guaranteed automatically by the definition of $\tilde{G}$.

Case 2: $G_j \bigcap \tilde{G}_j \neq G_j$ for at least one $j = 1, 2$. In this case, the separation condition (22) assures that we can take $\breve{G}_j$ with $\breve{G}_1 \bigcap \breve{G}_2 = \emptyset$ such that $(G_j \bigcup \tilde{G}_j) \subset \breve{G}_j$ with $j = 1, 2$. Now we have the far-field pattern decomposition

$$u^\infty = \breve{u}_1^\infty + \breve{u}_2^\infty$$

in terms of $\breve{G} := \breve{G}_1 \bigcup \breve{G}_2$ as well as $\breve{u}_j^s(x)$ outside $\breve{G}_j$ constructed in terms of $\breve{\varphi}_j$. Noticing the facts $G_j \bigcap \breve{G}_j = G_j\ \tilde{G}_j \bigcap \breve{G}_j = \tilde{G}_j$, using the uniqueness result in Case 1, we get that

$$u_j^s(x) = \breve{u}_j^s(x), \tilde{u}_j^s(x) = \breve{u}_j^s(x),\ x \in \mathbb{R}^m \setminus \overline{\breve{G}_j},$$

which implies that

$$u_j^s(x) = \tilde{u}_j^s(x),\ x \in \mathbb{R}^m \setminus \overline{\breve{G}_j}. \tag{25}$$

Since both $u_j^s(x)$ and $\tilde{u}_j^s(x)$ are analytic in $\mathbb{R}^m \setminus \overline{G_j \bigcup \tilde{G}_j}$ and $\breve{G}_j \supset G_j \bigcup \tilde{G}_j$, (25) yields that

$$u_j^s(x) = \tilde{u}_j^s(x),\ x \in \mathbb{R}^m \setminus \overline{G_j \bigcup \tilde{G}_j}$$

from the unique continuation of analytic function, which proves (21). Also, (21) leads to (20) immediately. This completes the proof. □

2.3 Far field splitting via Green's formula

We now establish an alternative splitting via Green's formula. We need the representation formula for both exterior and interior problems as follows. For some solution u to the Helmholtz equation in a domain G_j we observe that ([5], Theorem 2.1)

$$\int_{\partial G_j} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} \Phi(x,y) \right\} ds(y) = \left\{ \begin{array}{ll} -u(x) & x \in G_j \\ 0 & x \notin G_j. \end{array} \right. \tag{26}$$

On the other hand, the radiating solution u^s to the Helmholtz equation in the exterior of G_j has the representation ([5], Theorem 2.4)

$$\int_{\partial G_j} \left\{ u^s(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu(y)} \Phi(x,y) \right\} ds(y) = \begin{cases} 0 & x \in G_j \\ u^s(x) & x \notin G_j. \end{cases} \tag{27}$$

We will use these formulas to derive a general splitting procedure which does not need to avoid interior eigenvalues of the domain G_j.

Using the potential operators, Green's formula for the radiating solution u^s of the Helmholtz equation can be written in the form

$$u^s = K u^s - S \frac{\partial u^s}{\partial \nu} \quad \text{in } \mathbb{R}^m \setminus \overline{G}. \tag{28}$$

Then, the normal derivative $\frac{\partial u^s}{\partial \nu}$ on $\Lambda \subset \partial G$ for $u^s(x)$ outside G meets

$$\frac{\partial u^s}{\partial \nu} = T u^s - K' \frac{\partial u^s}{\partial \nu} \quad \text{in } \partial G \tag{29}$$

due to the jump relation of potential functions. This equation is not adequate to calculate the normal derivative from its boundary values, since for interior eigenvalues for the negative Lapcacian it lacks uniqueness and thus existence for general boundary values. For this reason we use the following operator representation of the the Dirichlet-to-Neumann map $B : u^s|_{\partial G} \to \frac{\partial u^s}{\partial \nu}|_{\partial G}$ or Steklov-Poincare operator, respectively. Following [5], page 48, with some parameter $\eta > 0$, the operator is given by

$$B := (i\eta I - i\eta K' + T)(I + K - i\eta S)^{-1} : C^{1,\beta}(\partial G) \to C^{0,\beta}(\partial G), \tag{30}$$

where one solves the exterior Dirichlet problem for the domain G by a Brackhage-Werner ansatz, also known as combined single- and double-layer potential, and calculates the normal derivative of the solution via the classical jump relations. In the following, we also need the far-field pattern of operator K which is defined as

$$(K^\infty \varphi)(\hat{x}) := \gamma \int_{\partial G} \frac{\partial e^{i\kappa \hat{x} \cdot y}}{\partial \nu(y)} \varphi(y) ds(y), \quad \hat{x} \in \mathbb{S}. \tag{31}$$

We now derive a splitting procedure via Green's formula.

Algorithm 2.7. *A splitting of the far field pattern u^∞ or the scattered field u^s in the exterior of $G = G_1 \cup G_2$ is obtained as follows.*

1. *Solve the integral equation*

$$u^{\infty} = (K^{\infty} - S^{\infty} B)\varphi \tag{32}$$

to obtain the boundary values $\varphi = u^s$ *on* ∂G *via the solution* $\varphi \in C^{1,\beta}(\partial G)$, *noticing (27).*

2. *Use (30) to evaluate the Dirichlet-to-Neumann map*

$$\psi := B\varphi \tag{33}$$

to calculate the normal derivative $\psi = \partial u^s/\partial \nu$ *on* ∂G *of the field* u^s *outside* G *in terms of (29).*

3. *Compute*

$$u_1^{\infty} = K_1^{\infty}\varphi_1 - S_1^{\infty}\psi_1, \qquad u_2^{\infty} = K_1^{\infty}\varphi_2 - S_2^{\infty}\psi_2, \tag{34}$$

$$u_1^s = K_1\varphi_1 - S_1\psi_1, \qquad u_2^s = K_2\varphi_2 - S_2\psi_2, \tag{35}$$

respectively to obtain a splitting of (u^{∞}, u^s), *where* $(S_j, K_j, S_j^{\infty}, K_j^{\infty})$ *with* $j = 1, 2$ *are defined by the same way as* $(S, K, S^{\infty}, K^{\infty})$ *with* ∂G *replaced by* ∂G_j.

We formulate the unique solvability and validity of Algorithm 2.7 in the following results.

THEOREM 2.8. *The equation (32) has a unique solution* φ *in* $C^{1,\beta}(\partial G)$ *for* $\beta \in (0, 1)$.

Proof. Since u^{∞} is the far-field pattern of scattered wave u^s outside D, the existence of $\varphi = u^s|_{\partial G} \in C^{1,\beta}(\partial G)$ satisfied (32) is obvious, noticing $u^s(x)$ is analytic in $\mathbb{R}^m \setminus \overline{D}$. Now let us consider the uniqueness. Assume that $\varphi(y) \in C^{1,\beta}(\partial G)$ satisfies

$$(K^{\infty} - S^{\infty} B)\varphi(\hat{x}) = 0, \quad \hat{x} \in \mathbb{S}.$$

Then $u^s(x)$ related to the far-field pattern given by (32) is identically zero outside G from Rellich lemma, that is,

$$(K - SB)\varphi(x) = 0, \quad x \in \mathbb{R}^m \setminus \overline{G}. \tag{36}$$

By defining $v_i^s(x) := (K_i\varphi_i - S_i(B\varphi)_i)(x)$ for $x \in \mathbb{R}^m$ with $i = 1, 2$, the above equation reads as

$$v_1^s(x) + v_2^s(x) \equiv 0, \quad x \in \mathbb{R}^m \setminus \overline{G}.$$

By the same argument as that in the proof of Theorem 2.2, we get that

$$v_j^s(x) \equiv 0, \quad x \in \mathbb{R}^m \setminus \overline{G}_j, \quad j = 1, 2.$$

On the other hand, $v_1^s(x)$ is the radiation solution outside G_1, it follows from (27) that $v_1^s(x) \equiv 0$ in G_1. Since both double-layer potential $K_1\varphi_1(x)$ and single-layer potential $S_1 B\varphi_1(x)$ solve the Helmholtz equation in $\mathbb{R}^m \setminus \partial G_1$, using the jump relation of K_1 and the continuity of S_1 with on ∂G_1 in the continuous density setting, we finally get from $v_1^s(x) \equiv 0$ in $\mathbb{R}^m \setminus \partial G_1$ that $\varphi_1(y)|_{\partial G_1} \equiv 0$. Similarly, we get $\varphi_2(y)|_{\partial G_2} \equiv 0$. □

Here we decompose the far-field pattern by Green formula, where $u^s|_{\partial G}$ is considered as the density. Comparing the decomposition of far-field pattern by general potential theory method in the previous section, the advantage of wave splitting based on Green formula is that we get u^s as well as its normal derivative directly. This kind of technique has been used in the reconstruction of Neumann data from the far-field pattern [9], such data are needed in the probe method [3]. Moreover, the wave splitting based on the Green formula does not need the condition of κ^2 not being the Dirichlet eigenvalue in G_j. But in the splitting based on general potential theory, we need this assumption, see the proof of Theorem 2.4.

We can also show the uniqueness of scattered wave splitting related to this far-field decomposition.

Theorem 2.9. *For given G, assume that the far field pattern u^∞ has the splitting $u^\infty = u_1^\infty + u_2^\infty$ in terms of (34). Then $u_i^s(x)$ defined in terms of (35) is the scattered wave in $\mathbb{R}^m \setminus \overline{G}_i$ corresponding to far-field pattern u_i^∞. Moreover, $u^s = u_1^s + u_2^s$ is the unique scattered wave splitting outside G related to u^∞ in the sense of Theorem 2.2.*

Proof. Obviously, $u_i^s(x)$ defined in terms of (35) is the radiating solution to the Helmholtz equation in $\mathbb{R}^m \setminus \overline{G}_i$. On the other hand, it follows from the relation between the operators $(K^\infty_{\partial G_i}, S^\infty_{\partial G_i})$ and $(K_{\partial G_i}, S_{\partial G_i})$ that $u_i^s(x)$ has the asymptotic expression with $u_i^\infty(\hat{x})$ given by (34). Therefore the Rellich lemma says that $u_i^s(x)$ must be the scattered wave outside G_i corresponding to far-field pattern u_i^∞. Finally, Theorem 2.2 generates the uniqueness of splitting outside $\overline{G}$, since $u_i^s(x)$ constructed here also satisfies the requirements in Theorem 2.2. □

In this subsection, we can also consider the analogy to Theorem 2.6, that is, the unique decomposition with respect to G based on the Green formula. It can be set up by the same way as that in the proof of Theorem 2.6, since (35) also splits the scattered wave in terms of the density functions $(u_i^s|_{\partial G_i}, \frac{\partial u_i^s}{\partial \nu}|_{\partial G_i})$ corresponding to the far-field decomposition (34). So we omit this result.

By the above theorem we can calculate u_j^s outside of the domains G_j. In $G_j \setminus \overline{D}_j$ we will show below that the scattered wave u_j^s can be calculated from u_j^∞ via *point source method*. We combine these two methods to calculate the total wave

$$u = u^i + u^s = u^i + u_1^s + u_2^s$$

around each obstacle D_j, $j = 1, 2$. Then we can use the zero points set of u to construct the boundary ∂D_j.

2.4 Determination of splitting domains via the range test

So far we have used the assumption that we know two domains G_1 and G_2 which contain the two components D_1 and D_2 of a scatterer D with the important condition $G_1 \cap G_2 = \emptyset$. Here we will discuss how these domains can be determined from the knowledge of the far field pattern u^∞ from one scattered time-harmonic wave. We will employ the *range test* as suggested by Kusiak, Potthast and Sylvester [12].

The range test exploits solvability arguments for the equation (14). Consider the equation in dependence of the unknown domain $G = G_1 \cup G_2$. Then we have the following result proven in [12].

THEOREM 2.10. *If the scattered field u^s defined by its far field pattern u^∞ can be analytically extended into the set $\mathbb{R}^m \setminus G$, then the far field equation*

$$S^\infty \varphi = u^\infty \tag{37}$$

does have a solution in $L^2(\partial G)$. In this case, u^s expressed in terms of the density φ can be extended to $\mathbb{R}^m \setminus \overline{G}$. If the field cannot be analytically extended into $\mathbb{R}^m \setminus \overline{G}$, then the equation (37) does not have a solution.

The solvability of the equation (37) can be numerically tested by calculating the regularized Tikhonov solution

$$\varphi_\alpha := (\alpha I + S^{\infty,*} S^\infty)^{-1} S^{\infty,*} u^\infty \tag{38}$$

and observing the behaviour of the norm $||\varphi_\alpha||_{L^2(\mathbb{S})}$ for $\alpha \to 0$. The key ingredient is Theorem 3.7 of [12] adapted to our notation.

THEOREM 2.11. *If the scattered field u^s defined by its far field pattern u^∞ can be analytically extended into the set $\mathbb{R}^m \setminus G$, then*

$$||\varphi_\alpha||_{L^2(\mathbb{S})} < \infty, \quad \alpha \to 0. \tag{39}$$

On the contrary, if the field cannot be analytically extended into $\mathbb{R}^m \setminus \overline{G}$, then

$$||\varphi_\alpha||_{L^2(\mathbb{S})} \to \infty, \quad \alpha \to 0. \tag{40}$$

The range test takes a reference setup $G = G_1 \cup G_2$ which incorporates some apriory knowledge about the possible location and size of the domains D_1 and D_2 such that some rotated and translated version of G can contain D_1 in G_1 and D_2 in G_2. Then, for every rotation and translation via some parameters ψ, θ and τ we test the solvability of the equation (37) via the calculation of the norm of (38). We use the setting for splitting for which the norm is minimal. For more details about the range test we refer to [13] and the literature cited therein.

3 Application of wave splitting for shape reconstruction

Based on the wave decomposition, we can reconstruct multiple scatterers by the following splitting procedure using a potential splitting and the point source method for field reconstructions. Here, for clarity of the presentation, the algorithm is explained for two obstacles.

ALGORITHM 3.1. *The reconstruction of D_1, D_2 by wave splitting:*

1. *First we specify two domains G_1 and G_2 such that $D_1 \subset G_1$ and $D_2 \subset G_2$, $G_1 \cap G_2 = \emptyset$ for unknown obstacles D_1, D_2. The possibility of this kind of choice depends on some a priori information about the location of D_1, D_2 as well as the extent of separation between D_1 and D_2.*

2. *Use the splitting procedure to determine densities φ_1 and φ_2 on ∂G_1 and ∂G_2, such that the single-layer potentials given by (15) generates a splitting of the scattered wave $u^s = u^s_1 + u^s_2$ and calculate the far field patters u^∞_1 and u^∞_2 in terms of φ_1 and φ_2. If the splitting is based on Green formula, then find density (φ_i, ψ_i) and get the expression (35) as well as the far-field decomposition (34).*

3. *Use the point source method to reconstruct u^s_2 from u^∞_2 in $G_2 \backslash \overline{D}_2$. We calculate an approximation $u^s_{2,\alpha}$ using appropriate masking operations which determine the illuminated area as suggested by Erhard [6].*

4. *Calculate an approximation* $u_{(\alpha)}$ *to the total field* u *in* $G_2 \setminus \overline{D}_2$ *by adding* u_1^s *and the incident field* u^i

$$u_{(\alpha)} = u^i + S_1\varphi_1 + u^s_{2,\alpha} \tag{41}$$

in case of splitting by potential theory, or

$$u_{(\alpha)} = u^i + K_1\varphi_1 - S_1\psi_1 + u^s_{2,\alpha} \tag{42}$$

with $\psi_1 = B\varphi_1$*, if the wave splitting is based on the Green formula.*

5. *Search for the zero curve of* u_α *to calculate an approximation to the boundary* ∂D_2*, provided that the component* D_2 *has the sound-soft type boundary.*

6. ∂D_1 *can be reconstructed analogously.*

Obviously the above shape reconstruction scheme can be applied to multiple obstacle with other kinds of boundary conditions on each component of D.

In the remaining part of this section we will give more details and a convergence analysis of step 3, that is, reconstruction of u_j^s, $j = 1, 2$, in $G_j \setminus \overline{D}_j$ from its far-field pattern by point source method. Notice, the expression (15) (or (35)) gives the scattered wave $u_j^s(x)$ only outside $\overline{G}_j$. Here we give the basic idea of the point source method based on potential theory as suggested by Liu, see [8, 11]. This approach to the point source method extends it to the reconstruction of general radiating fields, whereas the use of reciprocity relations as employed in [10] limits it to fields arising from scattering of plane waves.

Since in our boundary reconstruction problem, ∂D_2 is unknown, we try to approximate ∂D_2 by the zero-curve of total wave near ∂D_2. So in practice, we compute u_2^s outside some chosen domain $H_2 \supset D_2$. The initial specification of H_2 depends on some *a-priori* information of D_2. With some rough zero-curve of total wave outside H_2, we can shrink H_2 continuously to get a better reconstruction of ∂D_2.

Algorithm 3.2. *The point source method for the recovery of* u_2^s *in the known domain* $G_2 \setminus \overline{H}_2$ *for given* H_2 *uses the following steps.*

1. *Approximate the point source* $\Phi(\cdot, x)$ *for any fixed* $x \in G_2 \setminus \overline{H_2}$ *by a superposition of plane waves*

$$\Phi(y, x) = \int_{\mathbb{S}} e^{i\kappa y\cdot d} g_x(d)\,ds(d), \quad y \in \partial H_2. \tag{43}$$

2. *Express the scattered wave* $u_2^s(x)$ *outside* H_2 *as well as its far-field pattern in terms of the density function by*

$$u_2^s(x) = \int_{\partial H_2} \Phi(y,x)\rho(y)ds(y), \quad x \in \mathcal{R}^2 \setminus \overline{H}_2, \tag{44}$$

$$u_2^\infty(\hat{x}) = \gamma \int_{\partial H_2} e^{-i\kappa\hat{x}\cdot y}\rho(y)ds(y), \quad \hat{x} \in \mathbb{S}. \tag{45}$$

3. *By inserting (43) into (44) and exchanging the order of integral, it follows that*

$$u_2^s(x) = \frac{1}{\gamma}\int_{\mathbb{S}} u_2^\infty(-d)g_x(d)ds(d), \quad x \in G_2 \setminus \overline{H}_2 \tag{46}$$

in terms of (45), which reconstructs $u_2^s(x)$ *from its far-field pattern.*

The equation (43) for superposition density may not have an exact solution. So it must be solved by regularization technique to get an approximate solution $g_x^\alpha(\cdot)$ such that (43) holds approximately. Then $(u_2^s)^\alpha(x)$ generated from (46) in terms of g_x^α gives an approximation of $u_2^s(x)$.

In our procedure of generating u_2^∞ from total far-field u^∞ by wave splitting, the error is unavoidable. Therefore our computation formula in fact is

$$(u_2^s)^{\alpha,\delta}(x) := \frac{1}{\gamma}\int_{\mathbb{S}} (u_2^\infty)^\delta(-d)g_x^\alpha(d)ds(d), \quad x \in G_2 \setminus \overline{H}_2, \tag{47}$$

using the noisy data $(u_2^\infty)^\delta$. The error between the computational result $(u_2^s)^{\alpha,\delta}(x)$ and the exact field $u_2^s(x)$ may be estimated by the following result [8].

THEOREM 3.3. *Let* $(u_2^\infty)^\delta(\hat{x})$ *be the noisy data of far-field pattern* $u_2^\infty(\hat{x})$ *satisfying*

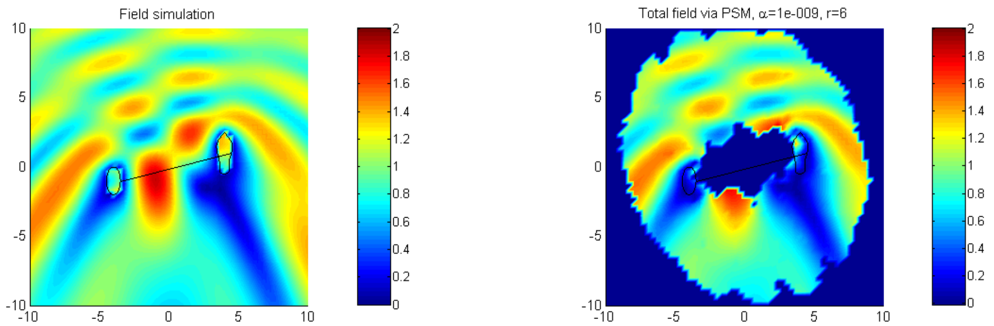
$$\|(u_2^\infty)^\delta(\cdot) - u_2^\infty(\cdot)\|_{L^2(\mathbb{S})} \leq \delta. \tag{48}$$

Then for any $x \in \mathcal{R}^2 \setminus \overline{\mathcal{H}(H_2)}$, *there exists constants* C, a, b, c *depending on* H_2, κ *and a choice strategy of* $\alpha = \alpha(\delta)$ *such that*

$$|(u_2^s)^{\alpha(\delta),\delta}(x) - u_2^s(x)| \leq C\delta^{\frac{1}{b\ln(-a\ln(c\delta))}} e^{-(-\ln\delta)^\beta} \tag{49}$$

with the constant C *uniformly in any compact set of* $\mathcal{R}^2 \setminus \overline{\mathcal{H}(H_2)}$, *where* $\mathcal{H}(H_2)$ *is the convex hull of* H_2.

We have shown all theoretical analysis on the wave splitting procedure related to multiple obstacle scattering. This procedure provides a flexible way to the wave computation and boundary recovery in inverse scattering. We will present some numerical results in next section to show the validity of this decomposition.



(a) (b)

Figure 1: (a) Simulation of the scattered field (b)Point source method using some circular approximation domain without splitting and without modifications which might take into account the non-convexity of the scatterer. The non-convex part of the fields and domains cannot be reconstructed since it is outside of the illuminated area of the method

4 Numerical examples

In this last section we demonstrate the feasibility of the splitting procedure by an application to the inverse acoustic scattering problem from two obstacles with Dirichlet boundary condition. For simplicity we restrict our attention to the two-dimensional case.

We have carried out a simulation of the wave scattering problem via a Brackhage-Werner potential approach

$$(50) \qquad u^s(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y)\, ds(y) - i \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^m \setminus \partial D,$$

leading to boundary integral equations of the second kind

$$(51) \qquad (I + K - iS)\varphi = -2u^i \quad \text{on } \partial D,$$

compare [5] or [10] for a detailed presentation. Employing Nystöm's method for the numerical solution of the integral equation and quadrature based on the trapezoidal rule the density potential can be evaluated on subsets of $\mathbb{R}^m$. Figures 1(a) and 2(a) show a plot of the modulus of the total field $u = u^i + u^s$ in a rectangle $Q = [-10, 10]x[-10, 10]$. The wave number has been chosen to $\kappa = 1$.

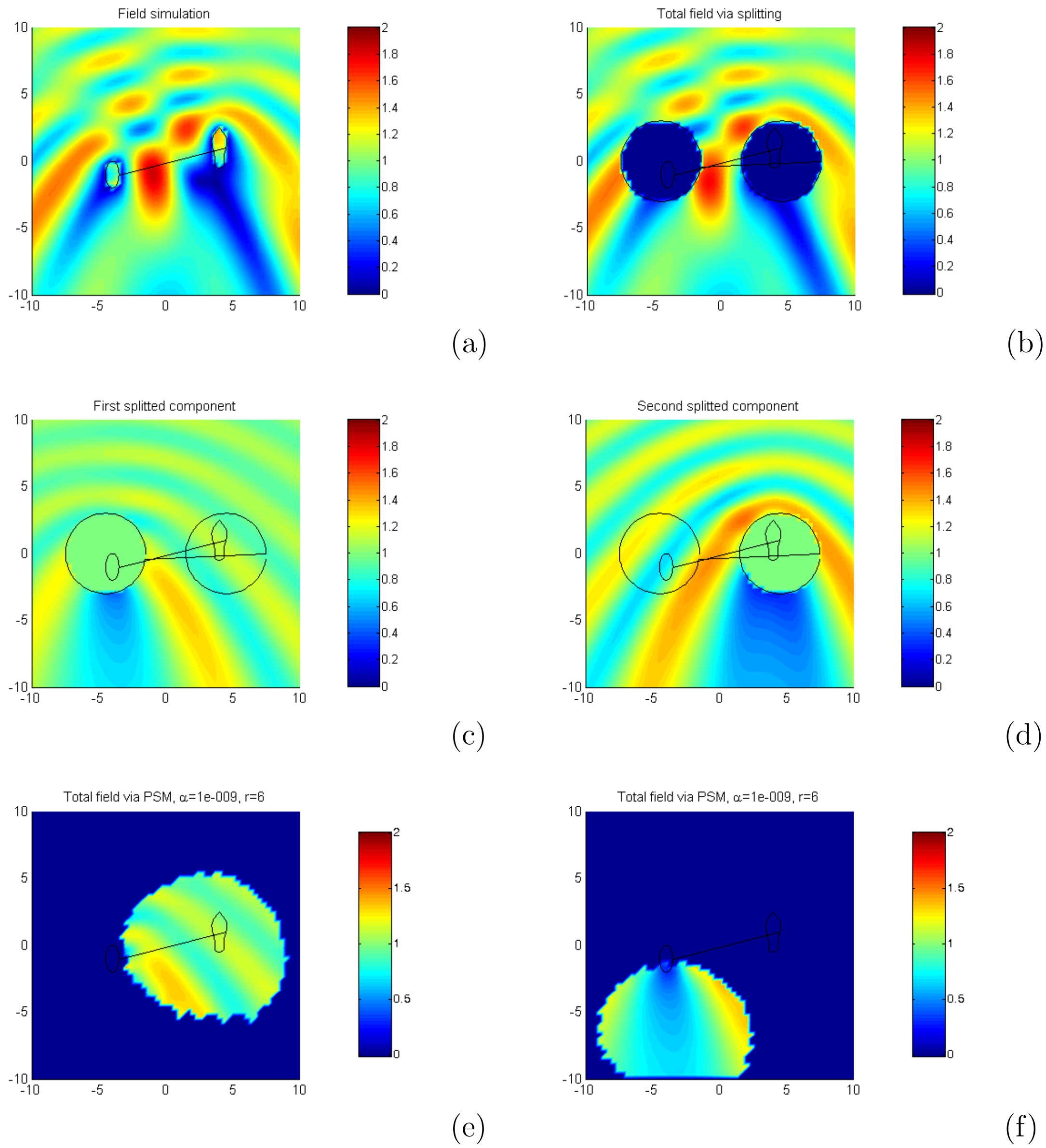
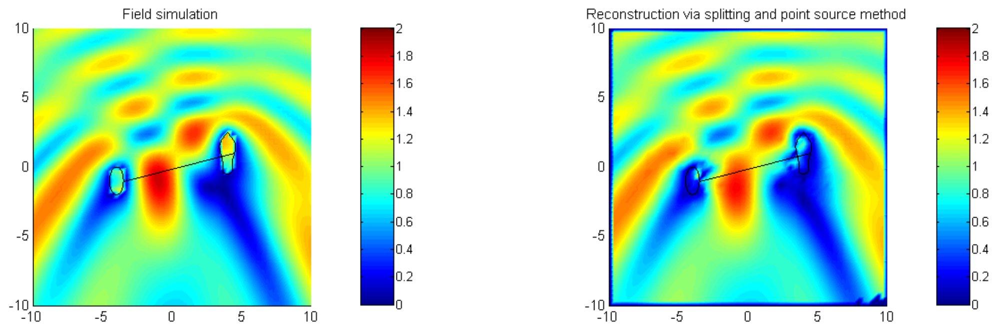


Figure 2: The images show the simulated field for scattering by two obstacles (a), the full reconstructed scattered field u^s via the splitting procedure with a single-layer approach following Algorithm 2.3 in figure (b), the field $u_1^s + u^i$ calculated via the splitting procedure in (c) and the field $u_2^s + u^i$ in (d). Reconstruction of $u_1^s + u^i$ from u_1^∞ on two illuminated areas around D_1 via the point source method is shown in (e) and (f). In particular, in (e) we obtain a reconstruction in an area where the point source method in its simple implementation cannot reconstruct the field.



(a) (a)

Figure 3: Original total field via simulation (a) and total field via splitting procedure and point source method after reconstruction (b).

First, we would like to demonstrate what can happen when the point source method is applied to the two obstacles without appropriate modifications taking into account the strong non-convexity of the scatterer D. We have employed the illumination technique developed by Erhard [6] to find the illuminated areas of the method and to set the field in areas with bad reconstruction to zero. An example is shown in Figure 1. Here, only the outside of the circular or convex hull of the scatterer is illuminated by the reconstruction scheme. Fields inside of the convex hull of the scatterer cannot be reconstructed and the algorithm shows this as blue areas between the domains D_1 and D_2.

We will demonstrate how the splitting procedure in combination with the point source method can overcome the above difficulties. In Figure (b) we demonstrate the scattered field as represented via the splitting procedure. The fields u_1^s and u_2^s are shown in Figures 2 (c) and (d), their sum adds up to the original field u^s in the exterior of the auxilliary domain $G = G_1 \cup G_2$. Here, we employed the single-layer approach for the splitting procedure following Algorithm 2.3. The field plotted in Figure 2 (b) is the one given by equation (18).

Next, we employ the point source method applied to the far field patterns u_1^∞. The point source method step by step reconstructs the field on illuminated areas, which depend on parameters of the method (compare [10] or [6]). We show two reconstructions of the field $u_1^s + u^i$ in Figure 2, (e) and (f). Please observe in particular the image (e), in which the field is reconstructed on the non-convex part between the scatterers D_1 and D_2. This was not possible without the splitting procedure.

Finally, we combine all illuminated partial reconstructions via some simple masking operations into a full reconstruction of the total field in $\mathbb{R}^m \setminus D = \mathbb{R}^m \setminus (D_1 \cup D_2)$. This is shown in Figure 3. From the reconstructed total field we are able to find the shape of the domains D_1 and D_2 searching for points where $|u(x)|$ is zero or close to zero. Since this is along the lines of [10] we omit further details and point to the literature.

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