Periodic solutions for nonlinear dilation equations

Peter Grindrod

*Department of Mathematics and Centre for Advanced Computing and Emerging Technologies, University of Reading,
Whiteknights PO Box 220, Reading, RG6 6AX UK.*

Abstract

We consider a class of functional equations representing nonlinear dilation maps of the real line having an invariant interval bounded above by a fixed point. Necessary and sufficient conditions for the existence of periodic solutions demand that the maps satisfy an eigenproblem, with integer eigenvalues, for a certain nonlinear generalisation of Chebyschev's ordinary differential equation. Hence we obtain generalisations of Chebyschev polynomials, where the associated functional equation has periodic solutions of a related Hamiltonian system. The maps given by Chebyschev polynomials, and their cosine solution, correspond to the special simplest case when the Hamiltonian system is linear.

Key words: Functional equations, nonlinear dilations, periodic solutions, Hamiltonians
PACS: 02.30.Ks, 02.30.Hq

1 Introduction

Aczel's book [1] provides an historical and systematic treatment of the solution of functional equations, one of the oldest topics within mathematical analysis. In [2] a further comprehensive overview and development is given.

Functional equations are at the heart of many subjects, including the foundation and derivation of the rules probability theory (see [4] and the references therein). More recently the role of the Feigenbaum functional equation [5] derived through a renormalisation approach to the super stability of attractors for unimodal one dimensional maps, is the key to the universality of the period doubling route to chaos.

There has been a huge amount of interest in linear dilation equations, since these are satisfied by wavelets [6]: such functional equations are common. Here we will address nonlinear dilations: where some unknown function is mapped by a given nonlinear function, and is to be recovered modulo a suitable dilation of the independent variable. Methods appropriate to linear dilation equations are clearly inapplicable to such problems.

In [3] a particular form of nonlinear dilation equation is introduced, comparimetric equations, that have applications in the analysis of multiple images. In this paper we will consider a general class of equation, that include comparametric equations, and our initial approach to proving the existence of solutions is, initially at least, based on convergence within a Banach function space.

Specifically we shall consider certain solutions $\phi(x)$ of the nonlinear dilation equation

$$\phi(\lambda x) = F(\phi(x)), \phi(0) = 1,\ \phi'(0) = 0,\ \phi''(0) < 0,$$

where F is a given smooth mapping from the interval [-1,1] into itself, satisfying $F(1) = 1$ and $F'(1) > 1$, and λ is a positive real to be determined also. (In the appliactions given in [3] F must be trivilally redefined so as to [0,1] onto itself.)

We show that there may exist periodic solutions only when λ takes integer values. Moreover the class of functions, F, for which such periodic solutions are admitted is precisely those which satisfy a certain nonlinear generalisation of Chebyshev's equation, a second order ordinary differential equation. Such preriodic solutions can be used to generate nonperiodic solutions for the same function F, having different values for λ though relaxing the condition $\phi''(0) < 0$.

2 A functional equation

Let F be a smooth mapping from the interval [-1,1] onto itself, such that $F(1) = 1$ and $F'(1) > 1$, where $'$ denotes differentiation.

We consider smooth solutions $\phi : \mathbb{R} \to [-1, 1]$ satisfying the functional equation,

$$\phi(\lambda x) = F(\phi(x)), \quad \phi(0) = 1 \text{ and } \phi'(0) = 0, \tag{1}$$

for some real $\lambda > 0$.

If $\phi_0(x)$ is a solution for (1) with $\lambda = \lambda_0$ say, such that $\phi_0'(x) \sim x^q$ for some $q > 1$ and small x, then $\phi_1(x) = \phi_0(x^s)$ is also a solution of (1) for

$\lambda = \lambda_1 = \lambda_0^{1/s}$. So setting $s = 2/(q+1)$ we see that $\phi_1'(x) \sim x$ for small x, and hence $\phi_1''(0) \neq 0$, and is therefore strictly negative (since one is an upper bound).

Conversely if $\phi_1(x)$ is a solution of (1), for which $\lambda = \lambda_1$ say, and $\phi_1''(0) < 0$, then one may generate a one parameter family of solutions for (1), via

$$\phi(x) = \phi_1(x^r), \ \lambda = \lambda_1^{1/r}, \quad r \geq 0,$$

for which both $\phi'(0)$ and $\phi''(0)$ vanish.

Hence we shall asume that we seek solutions for (1) satisfying $\phi''(0) < 0$.

In addition any solution, $\phi(x)$, for (1), is determined up to a rescaling of the x axis (since for any constant, α, $\phi(\alpha x)$ is also a solution). Therefore without loss of generality we shall impose the scaling condition

$$\phi''(0) = -1. \tag{2}$$

Immediately it follows from (1) and (2) that ϕ has the Maclaurin expansion $\phi(x) = 1 - \frac{x^2}{2} + ...$

Differentiating in (1) twice with respect to x, and setting x to zero, we obtain the condition

$$\lambda^2 = F'(1) > 0.$$

For example, if $F(y) = T_n(y)$, the nth Chebyschev polynomial (see [7][8] and the references therein), then $\lambda = n$ and $\phi(x) = \cos(x)$, for all $n > 0$. In that case (1) corresponds to the well known formula $\cos(nx) = T_n(\cos x)$.

Let $F^{(m)}$ denotes the mth iterate of F. As $\lambda > 1$ it is sufficient to solve (1) on an interval about the origin, [-1,1] say, and employ $\phi(\lambda^m x) = F^{(m)}(\phi(x))$ to evaluate ϕ elsewhere.

Let us define a sequence of even functions in $C^\infty[-1,1]$, all taking values within [-1,1], by

$$\phi_0(x) = 1 - \frac{x^2}{2} \ \text{ and } \phi_{k+1}(x) = F(\phi_k(x/\lambda)), \ k = 1, 2 \ldots. \tag{3}$$

The following result guarantees a solution to (1) and ((2).

Theorem 1 *For $\lambda = \sqrt{F'(1)} > 1$ there exists an even solution of (1) and (2). Furthermore as $k \to \infty$ the sequence $\phi_k(x)$ in (3) converges uniformly on [-1,1] to such a solution.*

Proof It is straightforward to show by induction that $\phi_n(0) = 1$, $\phi'_n(0) = 0$, $\phi''_n(0) = -1$, and ϕ_n is even for all n. Therefore we show that the sequence converges: the rest follows immediately. Applying the mean value theorem

$$|\phi_{k+1}(x) - \phi_k(x)| = |F^{(k)}(F(\phi_0(\frac{x}{\lambda^{k+1}}))) - F^k(\phi_0(\frac{x}{\lambda^k}))|$$

$$= |\frac{dF^{(k)}}{dx}(\theta)||F(\phi_0(\frac{x}{\lambda^{k+1}})) - \phi_0(\frac{x}{\lambda^k})|,$$

for some θ between $\phi_0(x/\lambda^k) = 1 - \frac{x^2}{2\lambda^{2k}}$ and $F(1 - \frac{x^2}{2\lambda^{2(k+1)}})$. The first factor behaves like $F'(1)^k = \lambda^{2k}$ as $k \to \infty$; and second factor behaves like $F''(1)x^4/4\lambda^{4(k+1)}$ as $k \to \infty$. Hence

$$|\phi_{k+1}(x) - \phi_k(x)| \to 0$$

uniformly on [-1,1] and the result follows.

The curve $(y, F(y))$ remains within the box $[-1,1] \times [-1,1]$: yet $\phi(x)$ may be periodic or wandering. For example if $F(n) = T_n(y)$ then $\phi(x) = \cos(x)$ is 2π periodic. However next we show that cases such as these are nongeneric.

Suppose the solution ϕ is P-periodic, satisfying $\phi(x+P) = \phi(x)$ for all $x \in [0, P]$, with some minimal period P (ϕ is not periodic for any smaller period, P'). Then we have, for all x,

$$\phi(\lambda P + x) = F(\phi(P + x/\lambda) = F(\phi(x/\lambda)) = \phi(x).$$

Hence ϕ is also Q-periodic, where $Q = \lambda P > P$.

If λ is an integer, then this is trivial. If λ is not an integer, then set $S \in (0, P)$ to be the remainder

$$S = Q\mathrm{mod}(P).$$

Then there is an integer k such that, for all x,

$$\phi(x + S) = \phi(x + Q - kP) = \phi(x - kP) = \phi(x).$$

This contradicts the assumption that P is the minimal period. Hence we have the following.

Corollary 2 *An integer value for λ is a necessary condition for the existence of a periodic solution ϕ of (1) and (2).*

It is natural to ask what class of functions F may admit periodic solutions for

(1) and (2). For such a periodic solution, $\phi(x)$, this requires that

$$F(y) = \phi(n\phi^{-1}(y))$$

is well defined considering all branches of ϕ^{-1}. Next we give a sufficient condition on F.

Theorem 3 *Let $\phi(x)$ be a twice continuously differentiable periodic function with range $[\beta, 1]$ (for some constant $\beta < 1$), satisfying*

$$\phi(0) = 1, \text{ and } \phi'(0) = 0$$

together with the equation

$$\phi''(x) = \dot{G}(\phi(x))/2, \tag{4}$$

for some smooth nonnegative function $G : [\beta, 1] \to \mathbb{R}^+$, where $\dot{G}(w)$ denotes the derivative of $G(w)$ at w, and satisfying $\dot{G}(1) = -2$, $G(\beta) = G(1) = 0$.

Then for any integer n, if F and ϕ also satisfy (1), for $\lambda = n$, (and (2)) then F is the solution of the differential equation

$$\frac{n^2}{2}\dot{G}(F(y)) = G(y)\frac{d^2F}{dy^2}(y) + \frac{1}{2}\dot{G}(y)\frac{dF}{dy}(y), \tag{5}$$

$$F(1) = 1, \ \frac{dF}{dy}(1) = n^2.$$

Conversely, suppose that $G(w)$ is differentiable and positive on $(\beta, 1)$, with simple zeros at β and 1 and satisfies $\dot{G}(1) = -2$. Then if F and ϕ satisfy (5) and (4), together with the boundary conditions, then they also satisfy (1) and (2) with $\lambda = n$.

Proof Using $\phi'(x)^2 = G(\phi(x))$, the integral of (4), together with (1), and $\phi''(nx) = \dot{G}(F(\phi(x)))/2$, we may obtain directly:

$$\frac{d^2}{dx^2}\left(\phi(nx) - F(\phi(x))\right) = \frac{n^2}{2}\dot{G}(F(y)) - G(y)\frac{d^2F}{dy^2}(y) - \frac{1}{2}\dot{G}(y)\frac{dF}{dy}(y). \tag{6}$$

Hence if F and ϕ satisfy (1) and (2), then setting the right hand side to be zero, we see F is the solution of (5) on [β,1], subject to the given boundary conditions.

Conversely when $G(w)$ is given as specified, suppose F is the solution of (5) on [β,1], and ϕ solves (4); then (5) may be integrated directly. First, multiplying through by $\frac{dF}{dy}(y)$, we obtain

$$n^2 G(F(y)) = G(y)\left(\frac{dF}{dy}\right)^2.$$

If we write $F(y) = f(x)$ where $y = \phi(x)$, this last becomes

$$n^2 G(f) = \left(\frac{df}{dx}\right)^2.$$

which is a rescaled version of the equation $\phi'(x)^2 = G(\phi(x))$ (that is equivalent to (4)). Hence by inspection $f(x) = \phi(nx)$, so that $\phi(nx) = F(\phi(x))$ as required.

Remark. In the special case that $G(w) = 1 - w^2$, we have $\dot{G}(w) = -2w$, and (5) is the Chebyshev (linear) differential equation [7] [8]; whence $F(y) = T_n(y)$, whilst (4) implies $\phi(x) = \cos(x)$. Thus (5), which in general in nonlinear (via the $\dot{G}(F)$ term), is a natural generalisation of the Chebyshev equation; and for each integer n there exists a whole class for functions F for which there exists a periodic solution to (1) and (2).

Remark. As an example, suppose $\phi(x) = \cos(x) + \epsilon(3\cos(3x) - 5\cos(5x) + 2\cos(7x))$ for some small constant $\epsilon > 0$. Then ϕ is even, 2π-periodic and satisfies $\phi(0) = 1$, $\phi'(0) = 0$ and $\phi''(0) = -1$ (and $\phi(\pi) = -1$) for all ϵ. An examination of ϕ'^2 reveals that $G(\phi) = \phi'(x)^2$ is well defined on (-1,1) for all $0 \leq \epsilon < 1/48$. For all such value for ϵ, and each n, integer, we may obtain a function, F, for which ϕ is a solution to (1) and (2).

Remark. Consider the nonlinear difference equation starting out from some z_0 in [-1,1]: $z_{n+1} = F(z_n)$. Since [-1,1] is invariant for F, the sequence remains there. Of the many famous results for such one dimensional interations, the statement that"period three implies chaos" [9] is one of the most memorable and reflects the position of period three orbits appearing at the end of Sharkovsky's sequence [10], where there are orbits all all possible periodicities. Suppose F is such that there exists a periodic solution ϕ satisfying (1) and (2), of period P, say. This may be guaranteed by our theorem in the previous section. Necessarily $\sqrt{F'(1)} = \lambda = n$ an integer. For all integers m for any x we have $F^{(m)}(\phi(x)) = \phi(n^m x)$. So we set $z_0 = \phi\left(\frac{P}{n^m - 1}\right)$, and apply the nonlinera iteration. Directly it follows that $z_m = \phi\left(\frac{n^m P}{n^m - 1}\right) = \phi\left(\frac{P}{n^m - 1}\right) = z_0$. Hence we have an m-periodic orbit. Thus if F is such that a periodic solution exists then the corresponding iteration map is chaotic, having orbits of all possible periods embedded within its attractor within [-1,1].

References

[1] Aczel, J., Lectures on Functional Equations and their Applications, Academic Press, New York, 1966; and Dover, New York, 2006.

[2] M. Kuczma, B. Choczewski, R. Ger, Iterative functional equations. Encyclopedia of Mathematics and its Applications, Vol. 32, Cambridge University Press, Cambridge 1990.

[3] S. Mann, Comparametric equations with practical applications in quatigraphic image processing, IEEE Trasactions on Image Processing, 9, No. 8, pp 1389-1406, 2000.

[4] Jaynes, E.T. Probability Theory: The Logic of Science, Cambridge, 2003.

[5] Feigenbaum, M. J. Quantitative universality for a class of non-linear transformations, J. Stat. Phys. 19, 25-52, 1978.

[6] Strang, G., Wavelets and Dilation Equations: A Brief Introduction, SIAM Review, Vol. 31, No. 4, pp. 614-627, 1989.

[7] Abramowitz, M., and Stegun, I.A., eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York, 1965.

[8] Weisstein, E.W. "Chebyshev Polynomial of the First Kind." From MathWorld– A Wolfram Web Resource. http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html

[9] Li, T. Y., and J. Yorke, Period three implies chaos, American Mathematical Monthly, LXXXII, 985-92, 1975.

[10] Stefan, P., A theorem of Sharkovsky on the existence of periodic orbits of continuous endomorphisms of the real line, Comm. Math. Phys. 54, 237-248, 1977.